Midsemestral Examination
Algebra IV
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Q 1. If L/K is an algebraic extension and $\alpha, \beta \in L$ have the same minimal polynomial over K, show that there is an isomorphism from $K(\alpha)$ to $K(\beta)$ which takes α to β and is identity on K.

OR

Let L/K be an extension of degree m. Let $f \in K[X]$ be an irreducible polynomial of degree n where m and n are co-prime. Then, prove that f is irreducible as an element of $L[X]$.

Q 2. Let L/K be a finite, normal extension. Let $f \in K[X]$ be irreducible. Then, show that the irreducible factors of f in $L[X]$ have the same degree.

OR

Let L/K be an extension field where the characteristic is $p > 0$. Suppose $\alpha \in L$ is algebraic over K. If the minimal polynomial of α over K has only one root, then prove that $\alpha^{p^n} \in K$ for some $n > 0$.

Q 3. Determine with proof the splitting field of the polynomial $X^{11} - 1$ over $\mathbf{F}_3$.

OR

Let K be a field of prime characteristic p. Let $a \in K$ be not a p-th power in K. Then, prove that $X^p - a$ is irreducible in $K[X]$.

Q 4. Find the possible characteristics of K for which $X^4 + X + 1$ has multiple roots. In each case, find the multiple roots and their multiplicities.

Q 5. Let $\alpha \in \mathbf{C}$ such that $\alpha^3 - 3\alpha + 1 = 0$. Prove that $K = \mathbf{Q}(\alpha)$ is a Galois extension of $\mathbf{Q}$. Find its Galois group.
Hint: Consider $\alpha^2 - 2$.

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Q 6. Find the Galois group of $\mathbf{Q}(\sqrt[4]{2}, i)$ over $\mathbf{Q}$. Use the fundamental theorem of Galois theory to find the subfields of $\mathbf{Q}(\sqrt[4]{2})$.